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Second Lecture on Inequalities

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Last week we proved the famous AGM Inequality

AGM Inequality: Let $a_1, a_2, \cdots, a_n$ be positive real numbers. Then

$$\sqrt[n]{a_1 a_2 \cdots a_n} \leq \frac{a_1 + a_2 + \cdots + a_n}{n}$$

and equality holds if and only if all the as are equal, that is $a_1 = a_2 = \cdots = a_n$

This result can be applied in many clever ways.

Example 1. Suppose a_1, a_2, a_3 are positive real numbers and $a_1 + a_2 + a_3 = 1$. We want to find the minimum value of $\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}$.

Now (1):

$$S = \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} = (a_1 + a_2 + a_3)\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}\right)$$

$$= \frac{a_1}{a_1} + \frac{a_1}{a_2} + \frac{a_1}{a_3} + \frac{a_2}{a_1} + \frac{a_2}{a_2} + \frac{a_2}{a_3} + \frac{a_3}{a_3} + \frac{a_2}{a_3} + \frac{a_3}{a_3}$$

$$= 3 + \left(\frac{a_1}{a_2} + \frac{a_2}{a_1}\right) + \left(\frac{a_2}{a_3} + \frac{a_3}{a_2}\right) + \left(\frac{a_3}{a_1} + \frac{a_1}{a_3}\right)$$

Suppose we have positive real numbers a, b and we apply the AGM with $n = 2$ to $\frac{a}{b}$ and $\frac{b}{a}$. We get $\frac{\frac{a}{b} + \frac{b}{a}}{2} \geq \sqrt{\frac{a}{b} \cdot \frac{b}{a}} = 1$ with equality occurring only for $\frac{a}{b} = \frac{b}{a}$. Hence $\frac{a}{b} + \frac{b}{a} \geq 2$ with equality occurring only when $a = b$.

So, in the equation (1), $\frac{a_1}{a_2} + \frac{a_2}{a_1}$, $\frac{a_2}{a_3} + \frac{a_3}{a_2}$ and $\frac{a_3}{a_1} + \frac{a_1}{a_3}$ are all at least 2, so [2

$$S \geq 3 + 2 + 2 + 2 = 9.$$

Also, we get equality only when $a_1 = a_2 = a_3 = \frac{1}{n}$

By a similar argument, one can get a corresponding result for n numbers:

Theorem. Let $a_1, a_2, \ldots, a_n$ be positive real numbers with $a_1 + a_2 + \cdots + a_n = 1$. Then

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \geq n^2$$

and equality occurs precisely when $a_1 = a_2 = \cdots = a_n = \frac{1}{n}$

Example 2. Let $a_1, a_2, \ldots, a_n$ be positive real numbers and let $s = a_1 + a_2 + \cdots + a_n$. Prove that $S = \frac{a_1}{s-a_1} + \frac{a_2}{s-a_2} + \cdots + \frac{a_n}{s-a_n} \geq \frac{n}{n-1}$.

Solution. Notice that $\frac{a_i}{s-a_i} + 1 = \frac{a_i + s - a_i}{s - a_i} = \frac{s}{s-a_i}$.

The second fraction $\frac{s}{s-a_i}$ is "nicer" to work with than $\frac{a_i}{s-a_i}$, since the numerator does not depend on i.

So we add 1 to each term of S and get

$$S + n = \frac{s}{s-a_1} + \frac{s}{s-a_2} + \cdots + \frac{s}{s-a_n} = sT,$$

where $T = \frac{1}{s-a_1} + \frac{1}{s-a_2} + \cdots + \frac{1}{s-a_n}$.

In Example 1 and the Theorem, we had a list of fractions of the form $1/z$ and we made progress by multiplying the sum by the sum of corresponding zs. We try this here. [3

$T\left((s-a_1)+(s-a_2)+\cdots+(s-a_n)\right) =$
n + the sum of terms like $\frac{s-a_1}{s-a_2}+\frac{s-a_2}{s-a_1}$
or more generally $\frac{s-a_i}{s-a_j}+\frac{s-a_j}{s-a_i}$ (so they look like $\frac{a}{b}+\frac{b}{a}$ which we had in Example 1)
So each of those terms is at least 2 and there are $\frac{n(n-1)}{2}$ such terms.

$$\text{So } T\left((s-a_1)+(s-a_2)+\cdots+(s-a_n)\right) \geq n+2\,\frac{n(n-1)}{2} = n^2.$$

Now $(s-a_1)+(s-a_2)+\cdots+(s-a_n) =$
$ns-(a_1+a_2+\cdots+a_n) = ns-s$
$= (n-1)s.$

Hence $T \geq \frac{n^2}{(n-1)s}$.

But $S+n = sT$, so $S+n \geq \frac{n^2 s}{(n-1)s}$

so $S+n \geq \frac{n^2}{n-1}$, so $S \geq \frac{n^2}{n-1} - n$
$= \frac{n^2-n(n-1)}{n-1}$

Hence $S \geq \frac{n}{n-1}$, as required.

One can use a similar trick to prove

<u>Example 3</u> Let $a_1, a_2, \ldots, a_n$ be positive real numbers and $s = a_1 + a_2 + \ldots + a_n$.
Then
$$\frac{s-a_1}{a_1} + \frac{s-a_2}{a_2} + \ldots + \frac{s-a_n}{a_n} \geq n(n-1).$$

Another famous inequality is the Cauchy-Schwarz inequality. We first state it for expressions with three terms.

Suppose $x_1, x_2, x_3, y_1, y_2, y_3$ are real numbers and that at least one of the xs is not zero.
Then
$$x_1y_1 + x_2y_2 + x_3y_3 \leq \sqrt{x_1^2+x_2^2+x_3^2} \cdot \sqrt{y_1^2+y_2^2+y_3^2}$$

and equality occurs if and only if there is a real number $r \geq 0$ with $y_1 = rx_1$, $y_2 = rx_2$, $y_3 = rx_3$.

<u>Example 1</u> $x_1 = 3$, $x_2 = 5$, $x_3 = 1$, $y_1 = 2$, $y_2 = 3$, $y_3 = 4$.

$x_1y_1 + x_2y_2 + x_3y_3 = 25$, $x_1^2 + x_2^2 + x_3^2 = 35$, $y_1^2 + y_2^2 + y_3^2 = 29$
and the inequality says $25 \leq \sqrt{35 \times 29} = \sqrt{1015}$.

<u>Example 2</u> $x_1 = -1$, $x_2 = 2$, $x_3 = -3$, $y_1 = -5$, $y_2 = 1$, $y_3 = 1$

$x_1y_1 + x_2y_2 + x_3y_3 = 4$, $x_1^2 + x_2^2 + x_3^2 = 14$, $y_1^2 + y_2^2 + y_3^2 = 27$

and the inequality says $4 \leq \sqrt{14 \times 27} = \sqrt{378}$.

The general Cauchy-Schwarz inequality states that if $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ are real numbers and at least one of the xs is not zero, then $x_1y_1 + x_2y_2 + \cdots + x_ny_n \leq \sqrt{(x_1^2+x_2^2+\cdots+x_n^2)(y_1^2+y_2^2+\cdots+y_n^2)}$.

If $n=2$, it says that $x_1y_1+x_2y_2 \leq \sqrt{(x_1^2+x_2^2)(y_1^2+y_2^2)}$

But consider

$$
\begin{aligned}
(x_1^2+x_2^2)(y_1^2+y_2^2) &= x_1^2y_1^2 + x_1^2y_2^2 + x_2^2y_1^2 + x_2^2y_2^2 \\
&= (x_1y_1+x_2y_2)^2 + (x_1y_2-x_2y_1)^2 \\
&\geq (x_1y_1+x_2y_2)^2 .
\end{aligned}
$$

The general proof is included in the version of this lecture on the web.

The Cauchy-Schwarz can be applied in very clever ways.

<u>Example 1</u>. Suppose x_1, x_2, x_3, x_4 are real numbers. Take $y_1 = y_2 = y_3 = y_4 = 1$. Then the inequality says

$$x_1+x_2+x_3+x_4 \leq \sqrt{(x_1^2+x_2^2+x_3^2+x_4^2)(1^2+1^2+1^2+1^2)} = 2\sqrt{x_1^2+x_2^2+x_3^2+x_4^2}$$

Similarly $-(x_1+x_2+x_3+x_4) \leq 2\sqrt{x_1^2+x_2^2+x_3^2+x_4^2}$

Squaring, we get

$$(x_1+x_2+x_3+x_4)^2 \leq 4(x_1^2+x_2^2+x_3^2+x_4^2).$$

[If a, b are real numbers with $0 \leq a \leq b$, then $a^2 \leq b^2$ and here either $x_1+x_2+x_3+x_4 \geq 0$ or $-(x_1+x_2+x_3+x_4) \geq 0$.

More generally, we have

Example 2. Let $x_1, x_2, \ldots, x_n$ be real numbers.

Then $(x_1 + x_2 + \cdots + x_n)^2 \leq n(x_1^2 + x_2^2 + \cdots + x_n^2)$.

Example 3. Suppose $x_1, x_2, \cdots, x_n$ are real numbers.

Then $x_1 x_2 + x_2 x_3 + x_3 x_4 + \cdots + x_{n-1} x_n + x_n x_1 \leq$
$x_1^2 + x_2^2 + \cdots + x_n^2$

[So for $n = 3$, this says $x_1 x_2 + x_2 x_3 + x_3 x_1 \leq x_1^2 + x_2^2 + x_3^2$]

Proof. Put $y_1 = x_2, y_2 = x_3, \cdots, y_{n-1} = x_n, y_n = x_1$
and note that $y_1^2 + y_2^2 + \cdots + y_n^2 = x_1^2 + x_2^2 + \cdots + x_n^2$.

So $x_1 y_1 + x_2 y_2 + \cdots + x_n y_n \leq \sqrt{(x_1^2 + x_2^2 + \cdots + x_n^2)(x_1^2 + x_2^2 + \cdots + x_n^2)}$

$$= \sqrt{(x_1^2 + x_2^2 + \cdots + x_n^2)^2}$$

$$= x_1^2 + x_2^2 + \cdots + x_n^2.$$

Example 4. (Minkowski's Inequality).

Let $a_1, a_2, \ldots, a_n, b_1, \cdots, b_n$ be real numbers.

Then $\left(\sum_{i=1}^n (a_i + b_i)^2 \right)^{1/2} \leq \left(\sum_{i=1}^n a_i^2 \right)^{1/2} + \left(\sum_{i=1}^n b_i^2 \right)^{1/2}$.

Proof $\sum_{i=1}^n (a_i + b_i)^2 = \sum_{i=1}^n a_i^2 + 2 \sum_{i=1}^n a_i b_i + \sum_{i=1}^n b_i^2$

$$\leq \left(\left(\sum a_i^2 \right)^{1/2} \right)^2 + 2 \left(\sum_{i=1}^n a_i^2 \right)^{1/2} \left(\sum_{i=1}^n b_i^2 \right)^{1/2} + \left(\left(\sum b_i^2 \right)^{\frac{1}{2}} \right)^2$$

$$= \left(\left(\sum_{i=1}^n a_i^2 \right)^{1/2} + \left(\sum_{i=1}^n b_i^2 \right)^{1/2} \right)^2$$

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Next Question: What is the maximum of $n^{1/n}$ for a positive integer n.

Solution. Let $a_n = n^{1/n}$. We first compare a_n and $a_{n+1} = (n+1)^{1/(n+1)}$. Notice that $c = a_n^{n(n+1)} = n^{n+1}$ while $d = a_{n+1}^{n(n+1)} = (n+1)^n$.

We now compare c and d.

We have
$$\frac{d}{c} = \frac{(n+1)^n}{n^{n+1}} = \frac{n^n\left(1+\frac{1}{n}\right)^n}{n^n\, n} = \frac{\left(1+\frac{1}{n}\right)^n}{n}.$$

Consider
$$\left(1+\frac{1}{n}\right)^n = 1 + \frac{n}{n} + \frac{\frac{n(n-1)}{2}}{n^2} + \frac{\frac{n(n-1)(n-2)}{3!}}{n^3} + \cdots + \binom{n}{n}\frac{1}{n^n}$$
(using the binomial theorem)

For $n \geq 3$,
$$\left(1+\frac{1}{n}\right)^n \leq 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{n!}$$
$$< 1 + \underbrace{1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}}}_{GP}$$
$$< 3.$$

So $\frac{d}{c} < \frac{3}{n}$ for $n \geq 3$.

Hence $\frac{d}{c} < 1$ for $n > 3$, that is
$$a_{n+1}^{n(n+1)} < a_n^{n(n+1)} \text{ for } n > 3 \text{ and}$$
thus $a_{n+1} < a_n$ for $n > 3$. So maximum of $n^{1/n}$ for a positive integer n is the maximum of $1^{1/1}, 2^{1/2}, 3^{1/3}$ and this is $3^{1/3}$. ANS: $3^{\frac{1}{3}}$.

Proof of Cauchy-Schwarz Inequality (CS)

The inequality states that if $x_1, \dots, x_n, y_1, \dots, y_n$ are real numbers, then $x_1y_1 + x_2y_2 + x_3y_3 + \dots + x_ny_n \le$
$$\sqrt{(x_1^2+x_2^2+\dots+x_n^2)(y_1^2+y_2^2+\dots+y_n^2)}.$$

Lagrange's Proof. We will illustrate it first when $n=3$.
Observe that $(x_1^2+x_2^2+x_3^2)(y_1^2+y_2^2+y_3^2) = (x_1y_1+x_2y_2+x_3y_3)^2$
$$+ (x_1y_2-x_2y_1)^2 + (x_2y_3-y_3x_2)^2 + (x_3y_1-x_1y_3)^2$$
To see this, note that in multiplying out the first term on the right one gets $x_1^2y_1^2+x_2^2y_2^2+x_3^2y_3^2$ and cross terms

and the cross-terms have the form $2x_1y_1x_2y_2$, $2x_2y_2x_3y_3$ and $2x_3y_3x_1y_1$ and that these terms arise with a minus in front when the squares $(x_1y_2-x_2y_1)^2$, $(x_2y_3-y_3x_2)^2$ and $(x_3y_1-x_1y_3)^2$ are multiplied out. Observe also that all the terms $x_i^2y_j^2$ $(1\le i\le 3, 1\le j\le 3)$ all occur when the right-hand-side is multiplied out. Since the terms $(x_1y_2-x_2y_1)^2$, $(x_2y_3-x_3y_2)^2$ and $(x_3y_1-x_1y_3)^2$ are squares of real numbers, they are all nonnegative. Hence $(x_1^2+x_2^2+x_3^2)(y_1^2+y_2^2+y_3^2) \ge (x_1y_1+x_2y_2+x_3y_3)^2$ and the result follows.

In the general case, one observes that
$$(x_1^2+x_2^2+\dots+x_n^2)(y_1^2+y_2^2+\dots+y_n^2) = (x_1y_1+x_2y_2+\dots+x_ny_n)^2$$
+ all terms of the form $(x_iy_j-x_jy_i)^2$ with $1\le i<j\le n$ (the argument is essentially the same as the $n=3$ case) and then
$$(x_1^2+x_2^2+\dots+x_n^2)(y_1^2+y_2^2+\dots+y_n^2) \ge$$
$$(x_1y_1+x_2y_2+\dots+x_ny_n)^2$$
and the result follows.

A more illuminating proof can be provided using [9
vectors. For this version, a vector $\underline{v} = (v_1, v_2, \cdots, v_n)$ is a list of real numbers $(v_1, \cdots, v_n)$ [In 2-dimensions $n = 2$ and one can think of these as coordinates of points in the plane]. We add vectors as follows: If $\underline{v} = (v_1, \cdots, v_n)$ and $\underline{w} = (w_1, \cdots, w_n)$, then $\underline{v} + \underline{w} = (v_1 + w_1, v_2 + w_2, \cdots, v_n + w_n)$ and if a is a real number, $a\underline{v} = (av_1, av_2, \cdots, av_n)$. We define the <u>norm</u> or <u>length</u> of the vector $\underline{v}$ by

$$\|\underline{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

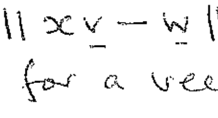
[In terms of coordinates, this is the distance of the point from the origin]. Note that $\|\underline{v}\| \geq 0$ and that $\|\underline{v}\| = 0$ happens only when <u>all</u> the numbers $v_1, \cdots, v_n$ are zero, since the square of a real number cannot be negative]. For $\underline{v} = (v_1, \cdots, v_n)$ and $\underline{w} = (w_1, \ldots, w_n)$, we define the <u>dot product</u> $\underline{v} \cdot \underline{w} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n$. The Cauchy–Schwarz inequality states that $\underline{v} \cdot \underline{w} \leq \|\underline{v}\| \cdot \|\underline{w}\|$.
For a real number x, we consider $\|x\underline{v} - \underline{w}\|^2$.
Note that $\underline{v} \cdot \underline{v} = v_1^2 + \cdots + v_n^2 = \|v\|^2$ for a vector $\underline{v}$. Now

$$\|x\underline{v} - \underline{w}\|^2 = (x\underline{v} - \underline{w}) \cdot (x\underline{v} - \underline{w})$$
$$= x^2 \underline{v} \cdot \underline{v} - 2x \underline{v} \cdot \underline{w} + \underline{w} \cdot \underline{w}$$

(Check this is true)

So $x^2 \underline{v} \cdot \underline{v} - 2x \underline{v} \cdot \underline{w} + \underline{w} \cdot \underline{w} \geq 0$.

So the graph of the function

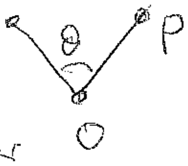


$$y = x^2 \underline{v} \cdot \underline{v} - 2x \underline{v} \cdot \underline{w} + \underline{w} \cdot \underline{w}$$

never goes below the x-axis. Solving the equation $x^2 \underline{v} \cdot \underline{v} - 2x \underline{v} \cdot \underline{w} + \underline{w} \cdot \underline{w} = 0$, by the standard formula, the term under the $\sqrt{\ }$ must be ≤ 0, so

$4(\underline{v} \cdot \underline{w})^2 \leq 4(\underline{v} \cdot \underline{v})(\underline{w} \cdot \underline{w})$ and this is the CS inequality.

There is a geometric interpretation of the Cauchy-Schwarz inequality. Just as in two-dimension, where each point in the plane can be described by a pair of numbers (x, y), namely its coordinates with respect to a chosen pair of axes, in n-dimensions, the same applies but we need n coordinates. So each point P corresponds to a list $\underline{x} = (x_1, x_2, \cdots, x_n)$ of real numbers. The distance $|OP|$ of P from the origin O (with coordinates $(0, 0, \cdots, 0)$) is $\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$. One can think of $\underline{x}$ as the vector $\overrightarrow{OP}$ or directed line segment from O to P.



If we have a point Q with corresponding coordinates $(y_1, \cdots, y_n)$ and (position) vector $\underline{y} = (y_1, y_2, \cdots, y_n)$, we can form a triangle OPQ and the cosine rule states that the lengths of the sides satisfy ⊛: $|PQ|^2 = |OP|^2 + |OQ|^2 - 2\,|OP|\,|OQ| \cos\theta$ where $\theta = \angle POQ$. For P with coordinates $(x_1, \cdots, x_n)$ and Q with coordinates $(y_1, \cdots, y_n)$, the distance

$|PQ| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$ and substituting this into ⊛ one gets

$$(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2 = (x_1^2 + x_2^2 + \cdots + x_n^2) + (y_1^2 + \cdots + y_n^2) - 2\|\underline{x}\|\,\|\underline{y}\| \cos\theta$$

Expanding out and cancelling terms and dividing by -2, we get $x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = \|x\|\,\|y\| \cos\theta$ and Cauchy-Schwarz follows from the fact that $-1 \leq \cos\theta \leq 1$. Thus $\dfrac{\underline{x} \cdot \underline{y}}{\|x\|\,\|y\|} = \cos\theta$, where θ is the angle between $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ at the origin.

Lectures on Inequalities

Solutions of the exercises

1. Find real numbers a, b, c with $a+b+c = 3$ and $abc = 10$ or prove that none exists.

<u>Answer</u>: The AGM inequality shows that there is no solution with a, b and c all positive. Start with taking one of them, c say, to be negative. Take $c = -10$, for example. Then we need real a, b with $ab = -1$ and $a+b = 13$. To solve, square the equation $a+b = 13$ to get $a^2 + 2ab + b^2 = 169$ and $4ab = -4$ subtracted, we get $(a-b)^2 = a^2 - 2ab + b^2 = 169 + 4 = 173$. Take $a - b = \sqrt{173}$. Solve $a+b = 13$ and $a - b = \sqrt{173}$ to get $2a = 13 + \sqrt{173}$ and $2b = 13 - \sqrt{173}$ (on adding and subtracting the equations). This yields

$$a = \tfrac{1}{2}(13 + \sqrt{173}), \quad b = \tfrac{1}{2}(13 - \sqrt{173}).$$

So one solution is $a = \frac{1}{2}(13 + \sqrt{173})$, $b = \frac{1}{2}(13 - \sqrt{173})$, $c = -10$.

There are lots of solutions, but one was just asked to find one.

2. Let n be an integer with $n \geq 2$. Prove that

$$\sqrt[n]{n} < 1 + \sqrt{\frac{2}{n}}.$$

<u>Answer</u>. Following the hint, let $a = \sqrt[n]{n} - 1$. Note that $a > 0$, since $n > 1$. Also $\sqrt[n]{n} = 1 + a$ and we take the nth power to get $n = (1+a)^n$ and now expand by the binomial Theorem. We get

$$n = 1 + \binom{n}{1} a + \binom{n}{2} a^2 + \cdots + \binom{n}{n} a^n$$
$$= 1 + na + \frac{n(n-1)}{2} a^2 + \cdots + a^n$$

Since $a > 0$ and all the binomial coefficients are positive, we obtain [12]

$$n \geq 1 + na + \frac{n(n-1)a^2}{2}$$
$$> 1 + \frac{n(n-1)a^2}{2}.$$

Hence

$$n-1 > \frac{n(n-1)a^2}{2}.$$

Since $n > 1$, $n-1 > 0$ and we can divide by it to get $1 > \frac{na^2}{2}$, that is, $a^2 < \frac{2}{n}$, and thus, since $a > 0$, $0 < a < \sqrt{\frac{2}{n}}$.

3. Let $x_1, \dots, x_n$ be positive real numbers with $\sum_{i=1}^{n} x_i = 1$ and let $x_{n+1} = x_1$ (NOT 1 as written on the Exercise sheet). Prove that

$$\sum_{j=1}^{n} \left(\frac{x_j^2}{x_j + x_{j+1}} \right) \geq \frac{1}{2}.$$

Answer: Following the hint $\frac{x_j^2}{x_j + x_{j+1}} = \frac{(x_j^2 + x_j x_{j+1}) - x_j x_{j+1}}{x_j + x_{j+1}}$

$$= \frac{x_j(x_j + x_{j+1}) - x_j x_{j+1}}{x_j + x_{j+1}} = \frac{x_j(x_j + x_{j+1})}{x_j + x_{j+1}} - \frac{x_j x_{j+1}}{x_j + x_{j+1}}$$

$$= x_j - \frac{x_j x_{j+1}}{x_j + x_{j+1}}.$$ Also $x_j x_{j+1} \leq \frac{(x_j + x_{j+1})^2}{4}$,

by the AGM inequality. Hence $\frac{x_j x_{j+1}}{x_j + x_{j+1}} \leq \frac{x_j + x_{j+1}}{4}$

and $x_j - \frac{x_j x_{j+1}}{x_j + x_{j+1}} \geq x_j - \frac{x_j + x_{j+1}}{4}$ (Note the $\geq$).

Hence

$$\sum_{j=1}^{n} \frac{x_j^2}{x_j + x_{j+1}} \geq \sum_{j=1}^{n} x_j - \sum_{j=1}^{n} \frac{x_j}{4} - \sum_{j=1}^{n} \frac{x_{j+1}}{4} = 1 - \frac{1}{4} - \frac{1}{4} = \frac{1}{2}$$

4. Let $n \geq 2$ be an integer. Prove that

$$\frac{1}{2} < \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n} < 1.$$

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ANSWER Part 1: We prove

$$\frac{1}{2} < \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n},$$

There are $n+1$ terms on the right hand side and the n terms $\frac{1}{n}, \frac{1}{n+1}, \ldots, \frac{1}{2n-1}$ are all greater than $\frac{1}{2n}$. So the right hand side is greater than

$$n \times \frac{1}{2n} + \frac{1}{2n} = \frac{1}{2} + \frac{1}{2n} > \frac{1}{2}.$$

Part 2: We prove

$$\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n} < 1.$$

There are $n+1$ terms on the lefthand side.
Suppose $n = 2$. Then the left-hand-side is $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{13}{12} > 1$, so the result is not true in this case.

Suppose $n \geq 3$. The third, fourth, $\ldots$, $(2n)$ th terms on the left-hand-side are $\leq \frac{1}{n+2}$, so the left-hand-side is $\leq \frac{1}{n} + \frac{1}{n+1} + \frac{n-1}{n+2}$

$$= \frac{n^3+2n^2+4n+2}{n(n+1)(n+2)}.$$

But $\frac{n^3+2n^2+4n+2}{n(n+1)(n+2)} = 1 - \left(\frac{n^2-2n-2}{n(n+1)(n+2)}\right)$ and

$n^2-2n-2 = (n-1)^2 - 3 > 0$ for $n-1 \geq 2$, that is $n \geq 3$.

Hence $\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n} < 1$ for $n \geq 3$.

5. Determine with proof which is bigger a or b in each of the following cases:

(i) $a = 7^{1/8}$, $b = 8^{1/7}$,

(ii) $a = \sqrt{101} - \sqrt{99}$, $b = \frac{1}{20}$,

(iii) $a = \sqrt{2} + \sqrt[3]{3}$, $b = \sqrt[4]{66}$,

(iv) $a = \log_2 3$, $b = \log_3 5$.

ANSWER (i) $(7^{1/8})^7 = 7^{7/8} < 7^1 = 7$, so $a < 7^{\frac{1}{7}}$. But $b = 8^{1/7}$. Hence $a < b$ in this case.

(ii) The trick here is to use $\sqrt{a} - \sqrt{b} = \dfrac{a-b}{\sqrt{a}+\sqrt{b}}$ for positive numbers a, b (To prove this, just cross-multiply).

$$a = \sqrt{101} - \sqrt{99} = \frac{101-99}{\sqrt{101}+\sqrt{99}} = \frac{2}{\sqrt{101}+\sqrt{99}} > \frac{2}{2\sqrt{101}}$$

(since $\sqrt{101} > \sqrt{99}$). Hence $a > \frac{1}{\sqrt{101}} > \frac{1}{\sqrt{10}} = \frac{1}{10}$)

so $a > b = \frac{1}{20}$.

(iii) For simplicity of calculation, put $\alpha = \sqrt[3]{3}$ and note that $\alpha^3 = 3$ and $\alpha^4 = 3\alpha$. Suppose for the sake of contradiction that $a = \sqrt{2} + \alpha < b = \sqrt[4]{66}$. Then $a^4 < b^4$, that is $(\sqrt{2}+\alpha)^4 < 66$, so

$$\alpha^4 + 4\alpha^3\sqrt{2} + 12\alpha^2 + 8\alpha\sqrt{2} + 4 < 66$$

and using $\alpha^3 = 3$, $\alpha^4 = 3\alpha$, this reduces to

$$3\alpha + 12\sqrt{2} + 12\alpha^2 + 8\alpha\sqrt{2} < 62,$$

so

$$\sqrt{2}(12 + 8\alpha) < 62 - 3\alpha - 12\alpha^2.$$

The left-hand-side is positive, so the right hand side is also, so we can square both sides (If p, q are real numbers with $0 < p < q$, then $p^2 < q^2$) to get $[\sqrt{2}(12+8\alpha)]^2 < (62 - 3\alpha - 12\alpha^2)^2$.

Multiplying out and collecting terms we get

$$3556 + 144\alpha^4 + 72\alpha^3 - 1607\alpha^2 - 756\alpha > 0$$

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and again using $\alpha^3 = 3$, $\alpha^4 = 3\alpha$, this becomes

① $$3772 > 324\alpha + 1607\alpha^2.$$

Since $\alpha > 0$, this implies

② $$3772\alpha > 324\alpha^2 + 1607\alpha^3 = 324\alpha^2 + 4821$$

(since $\alpha^3 = 3$). Now using ①

$$3772^2 > 324 \times \underbrace{3772\alpha} + 1607 \times 3772\alpha^2$$

and using ② this gives

$$3772^2 > 324(324\alpha^2 + 4821) + 1607 \times 3772\alpha^2.$$

Multiplying this out, we get

$$12665980 > 6166580\alpha^2$$

and multiplying by α, we get

$$12665980\alpha > 6166580\alpha^3 = 18499740$$

and $$\alpha > \frac{18499740}{12665980} \quad \ldots \text{③}$$

But $$\left(\frac{18499740}{1266598}\right)^3 = 3 + \frac{294325815972831 06}{25399572483222 9899}$$

$$> 3.$$

So $\frac{18499740}{12665980} > \sqrt[3]{3} = \alpha$. This contradicts ③. So our assumption $a < b$ is false.

Also if $a = b$, we would get equality in ③. So $a \neq b$. Hence $a > b$.

[Comment: Even though the calculations are very messy in this example, the basic idea is to keep squaring terms until we get rid of all the $\sqrt{\ }$ terms. One is then left with an inequality involving only terms in α and α^2 and various real numbers. The last step is to get a second inequality by multiplying by α and then using the two inequalities to get an inequality for α itself, cubing and comparing the result with $3 = (\sqrt[3]{3})^3$]

(iv) $a = \log_2 3$ and $b = \log_3 5$. This requires a different kind of trick to the others as we do not simplify the problem by squaring or cubing or steps like those in the last three examples.

Now $a = \log_2 3$ means $a > 0$ and $2^a = 3$.
Also $b = \log_3 5$ means $b > 0$ and $3^b = 5$.

Consider $2^{3/2}$. Note that $\left(2^{3/2}\right)^2 = 2^3 = 8 < 9 = 3^2$, so taking square roots here, we get $2^{3/2} < 3$. But $2^a = 3$. So $a > \frac{3}{2}$ ... ①

Next $\left(3^{3/2}\right)^2 = 3^3 = 27 > 25 = 5^2$, so taking square roots, $3^{3/2} > 5$. But $3^b = 5$. So $b < 3/2$ ... ②. Now ① and ② imply that $b < a$.